SPECTRAL TRANSFORMS

- **Introduction**
- **Feature Space**
- **Principal and Tasseled-cap Components**
- **Spectral Indices (already covered)**

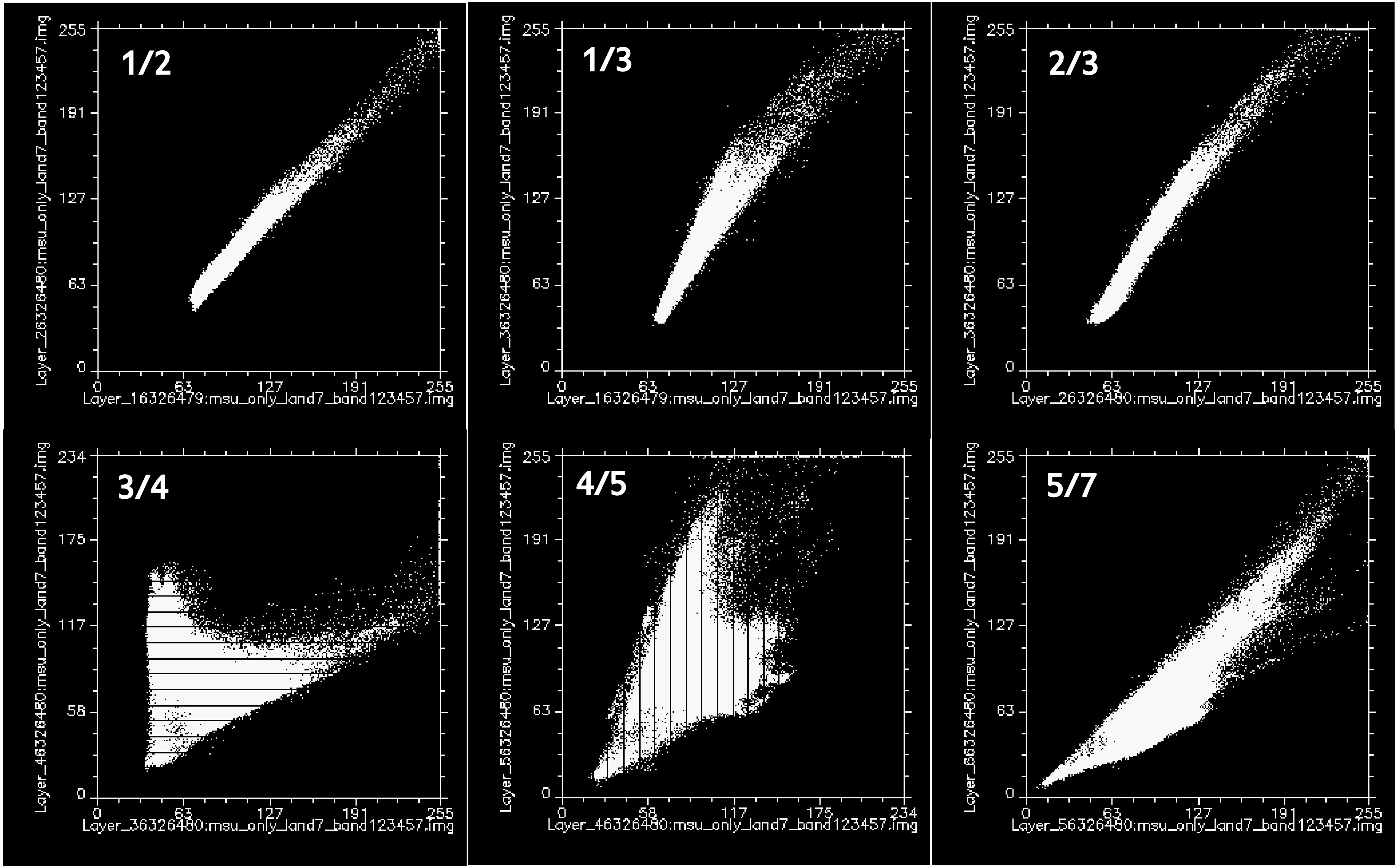
Introduction

- There are four “spaces” associated with multispectral remotely sensed images:
  - Spatial Space
    - the DN(x,y,z) space,i.e. an “image”
  - Spectral Space
    - K – D vector
  - Feature Space
    - a transformed image or spectral space
  - Temporal Space
    - A temporal vector space

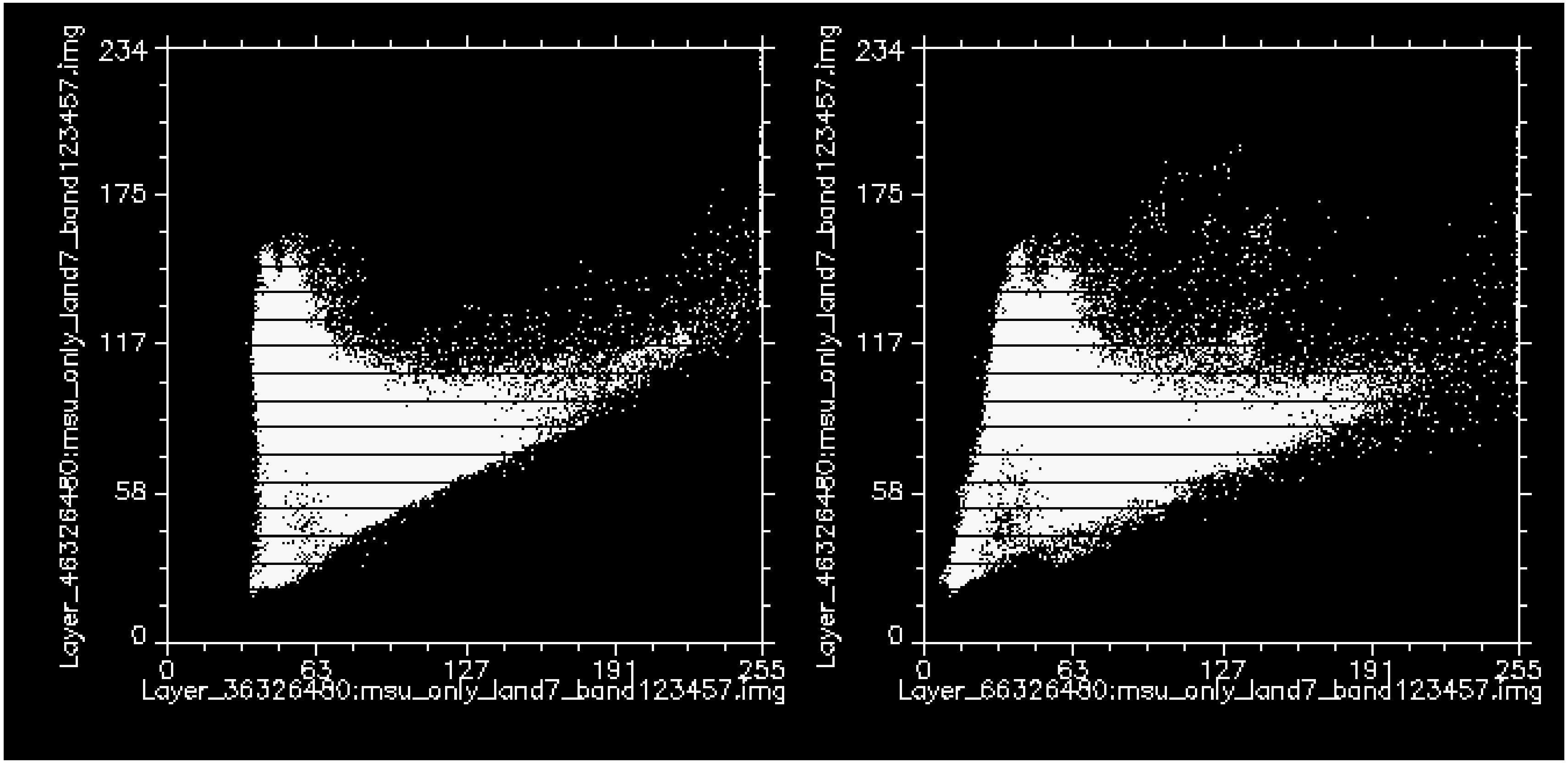
Spatial Space Spectral Space Feature Space Temporal Space

Feature Space

Spectral bands are often correlated



Spectral bands are often correlated

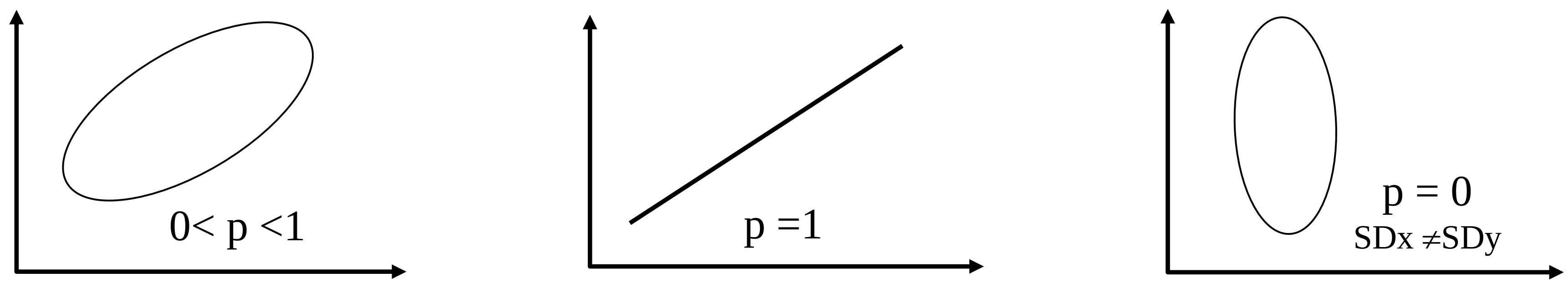


Transformation

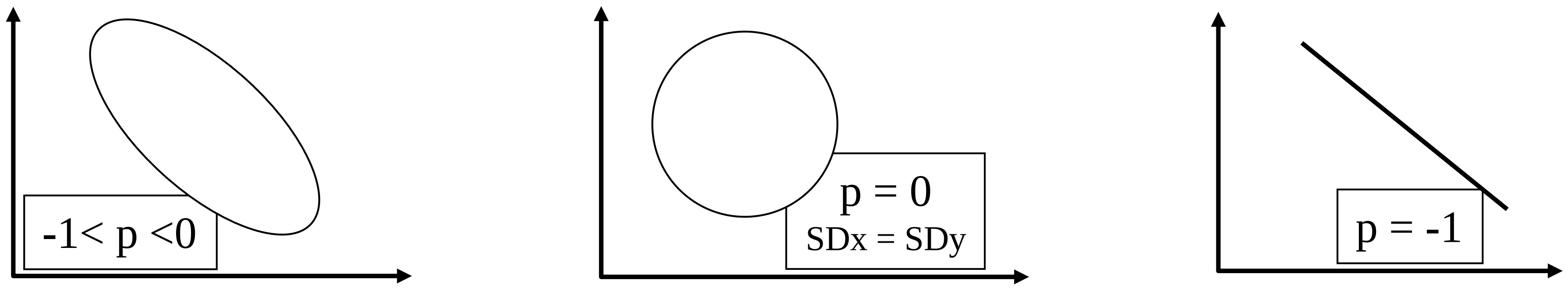
- Linear spectral transform
  - Corresponds to a coordinate rotation of the *DN* space to the *DN'* space
  - Example:principal components transform
- Nonlinear spectral transform
  - Example: multispectral ratios
- In either case,*DN'* is the derived *feature space*

Information Redundancy

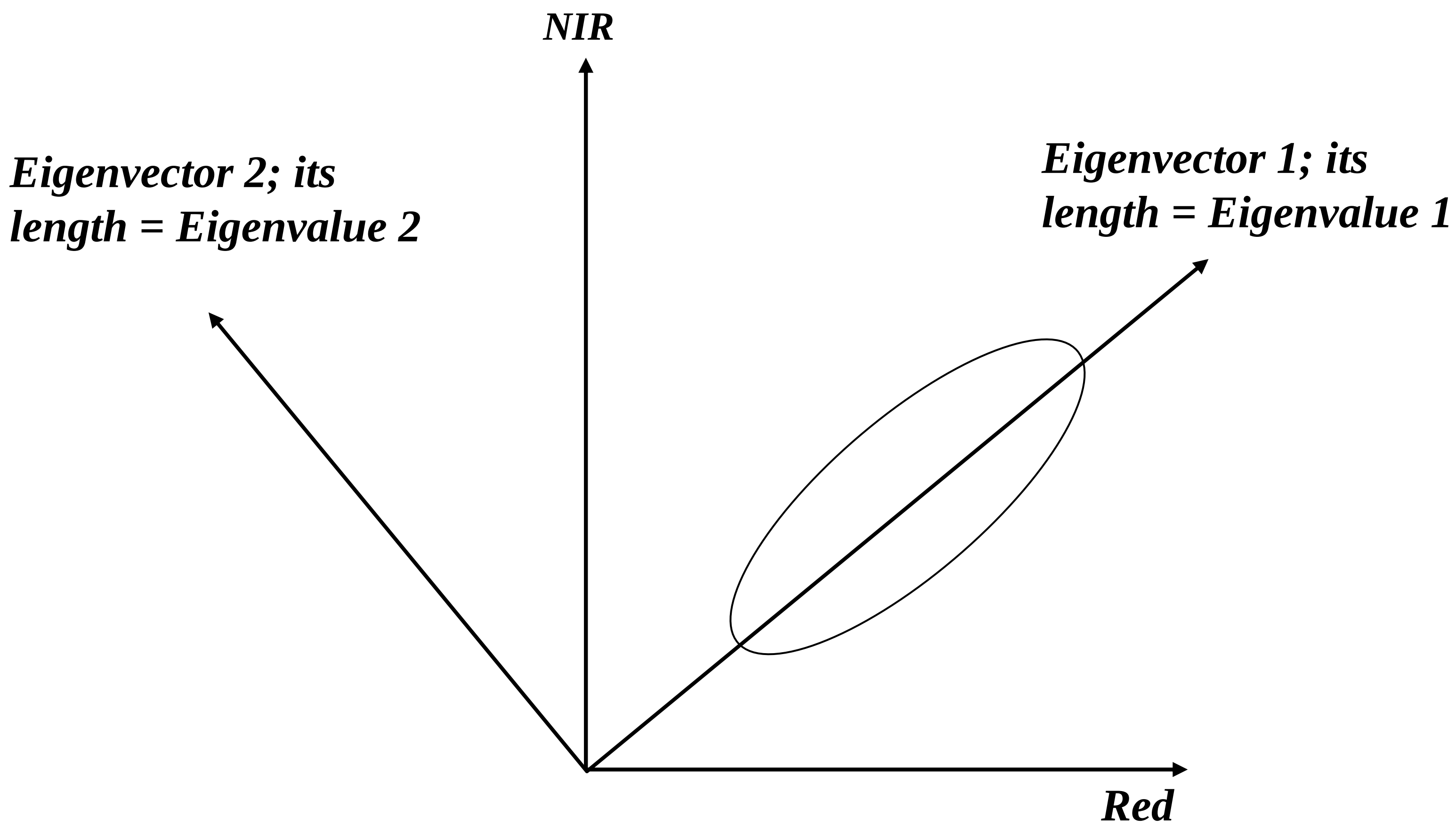
- Much of the information contained in multispectral bands such as those TM or ETM is redundant, i.e., the spectral bands are highly correlated:



Information Redundancy

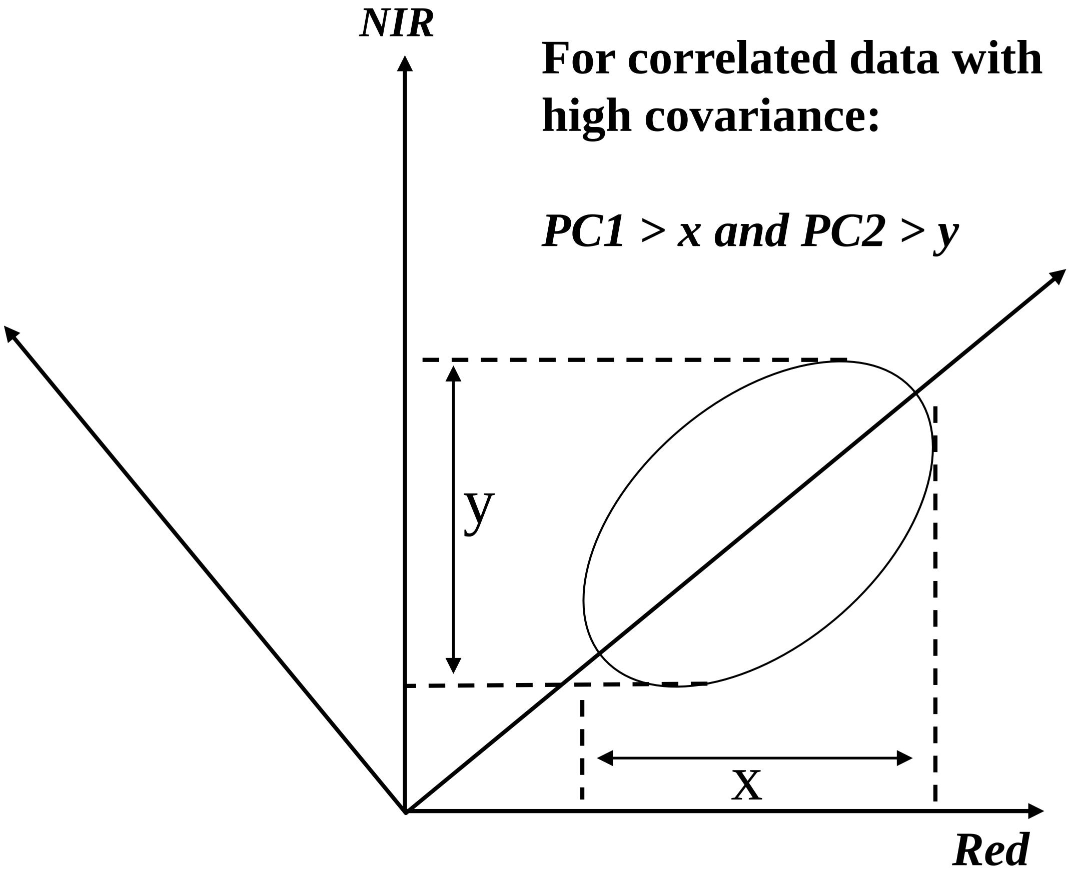


PCT Characteristics



- If a pre-labeled data is used to find uncorrelated axes-of-separation, the analysis is termed ***Canonical Analysis***

PCT Characteristics



For correlated data with high covariance:

PC1 > x and PC2 > y

PCT Characteristics

- The principal components are UNCORRELATED (i.e. orthogonal), even though the original variables are correlated
- Eigenvalues = total variance contained along each transformed axis (i.e. the length of the eigenvector)
- Eigenvectors = the individual transformed axes; they define the principal components directions
- The total variance of the original data set = the total variance of the transformed data set.

PCT Characteristics

- PCT is based on the variance and the covariance of the data set (i.e. data dependent)
- Variance = measure of the scatter or spread within one variable of a data set

$$VAR = \frac{n\sum x^2 - \left(\sum x\right)^2}{n^2}$$

- Covariance = measure of the scatter or spread between two variables of a data set

General Form of PCT

- *Principal Components Transform (PCT)*

  $PC = W_{PC} \times DN$

  - *Linear matrix transform*
  - *PC is the K–dimensional Principal Component (PC)vector*
  - *Each PCk is a weighted sum of all spectral bands*
  - W_{PC} *is a K x K transformation matrix*

PCT Properties

- *W_{PC} diagonalizes the covariance matrix, C, of the original image*

$$C_{PC} = W_{PC} C W_{PC}^T$$

  - *Since C_{PC} is diagonal, the PC components are uncorrected*

- *The diagonal elements of C_{PC} are the eigenvalues of the data*

PCT Properties

$$C_{PC} = \begin{bmatrix} \text{eigenvalue 1} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \text{eigenvalue K} \end{bmatrix} = \begin{bmatrix} \lambda_1 \ldots & 0 \\ \ldots \ldots & 0 \\ 0 \ldots & \lambda_K \end{bmatrix}$$

- *Each eigenvalue, λ_k ,is equal to the variance of the corresponding PC_k and is found by solving the characteristic equation.*

PCT Properties

- *W_{PC} consists of the eigenvectors of the data along its rows,*

$$W_{PC} = \begin{bmatrix} \text{eigenvector}_1^{\text{t}} \\ \vdots \\ \text{eigenvector}_{\text{K}}^{\text{t}} \end{bmatrix} = \begin{bmatrix} e_1^t \\ \vdots \\ e_K^t \end{bmatrix} = \begin{bmatrix} e_{11} & \dots & e_{1K} \\ \vdots & & \vdots \\ e_{K1} & \dots & e_{KK} \end{bmatrix}$$

- *Each eigenvector, e_k, consists of the weights applied to the original bands to obtain PC_k and is found by solving the equation,*

PCT Properties

- *It is a rigid rotation in K-D of the original coordinate axes to coincide with the major axes of the data*
- *Although the PC axes are orthogonal to each other in K-D, they are generally not orthogonal when projected to the original multispectral space*
- *It optimally redistributes the total image variance in the transformed data*
- *The transform, Wpc, is data dependent*
- *The PCT can also be applied to multitemporal datasets. However, interpretation is generally difficult, since you are dealing with the spectral-temporal variations*

PCT Theory

$$[Z]_o [C]_A = [?]\,[C]_A$$

where

$$[Z]_o = [D]^T [D]$$

PCT Theory

The response matrix is then:

$$[R]_A = [D]\,[C]_A^T$$

where $[C]^{-1} = [C]^T$

PCT Theory

	20%	40%	75%	
	14.13	10.31	4.72	$\lambda 1$
	20.92	16.51	9.07	$\lambda 2$
	25.65	17.88	5.65	$\lambda 3$
D =	41.78	45.75	51.98	$\lambda 4$
	45.32	45.49	42.70	$\lambda 5$
	42.03	35.52	22.99	$\lambda 6$
	34.46	24.51	9.33	$\lambda 7$

$Zo = [D]^T [D] =$

	8048.7	7260.3	5796.0
=	7260.3	6723.4	5665.2
	5796.0	5665.2	5277.3

PCT Theory

$$[R]_A = [R1\ R2\ R3] = \begin{matrix} 17.4076 & -5.0156 & 0.2231 & \lambda 1 \\ 27.5461 & -5.8055 & -0.0095 & \lambda 2 \\ 29.6507 & -11.4173 & 0.1469 & \lambda 3 \\ 79.4493 & 15.1036 & 0.4913 & \lambda 4 \\ 76.9122 & 5.5487 & -0.5012 & \lambda 5 \\ 59.1074 & -7.9304 & -0.3919 & \lambda 6 \\ 40.9911 & -13.9600 & 0.3587 & \lambda 7 \end{matrix}$$

Data decomposition for the first three components

Component (n)	Eigenvalue (l)	Variance (%)	RSD (+/-)	Eigenvector	(C)	
1	1934.3	96.475	7.11	0.6351	0.5895	0.4991
2	70.6	3.5208	0.35	-0.5908	-0.0455	0.8055
3	0.1	0	0	0.4976	-0.8065	0.3194

PCT Example

- Here is an example to show the variance invariant nature

Mean Band 4 = 129.29 (range 0-255)
Mean Band 5 = 106.63 (range 0-255)
Mean Band 6 = 104.71 (range 0-255)
Mean Band 7 = 121.80 (range 0-255)

Covariance Matrix:

$\sigma^2_{11} = 5192.16$	$\sigma^2_{12} = 3866.65$	$\sigma^2_{13} = 2722.83$	$\sigma^2_{14} = 1094.98$
	$\sigma^2_{22} = 3781.67$	$\sigma^2_{23} = 2520.50$	$\sigma^2_{24} = 1462.00$
		$\sigma^2_{33} = 4806.52$	$\sigma^2_{34} = 3652.66$
			$\sigma^2_{44} = 3927.78$

Sum of variance = 17708.13

PCT Example

- After the data is transformed with PCT

Eigenvalues and eigenvectors are:

$\lambda_1 = 12252.33$ $\quad e_1 = (+0.542, +0.483, +0.559, +0.401)$

$\lambda_2 = 4464.40$ $\quad e_2 = (-0.563, -0.356, +0.403, +0.627)$

$\lambda_3 = 639.82$ $\quad e_3 = (-0.370, +0.682, -0.504, +0.379)$

$\lambda_4 = 351.58$ $\quad e_4 = (+0.503, -0.418, -0.521, +0.549)$

Sum of variance = 17708.13

NOTE: the total variance remained the same!

PCT Demo

This document is to provide you with an example of calculating eigenvalues and eigenvectors for Principal Component Analysis.

If you have a dataset collected with a radiometer or from an image in, say, three spectral bands. The data in ASCII look like

$$D = \begin{bmatrix} 0.337 & 0.378 & 0.424 \\ 0.254 & 0.291 & 0.285 \\ 0.187 & 0.212 & 0.116 \\ 0.179 & 0.458 & 0.245 \\ 0.102 & 0.339 & 0.075 \\ 0.057 & 0.52 & 0.093 \\ 0.045 & 0.475 & 0.071 \\ 0.038 & 0.654 & 0.079 \\ 0.062 & 0.093 & 0.246 \end{bmatrix}$$

λ_1	λ_2	λ_3	Target
0.337	0.378	0.424	1
0.254	0.291	0.285	2
0.187	0.212	0.116	3
0.179	0.458	0.245	4
0.102	0.339	0.075	5
0.057	0.52	0.093	6
0.045	0.475	0.071	7
0.038	0.654	0.079	8
0.062	0.093	0.246	9

The following are steps to compute the eigenvalues and eigenvectors:

Step **1**: Transposed Data Matrix

$$D^T = \begin{bmatrix} 0.337 & 0.254 & 0.187 & 0.179 & 0.102 & 0.057 & 0.045 & 0.038 & 0.062 \\ 0.378 & 0.291 & 0.212 & 0.458 & 0.339 & 0.52 & 0.475 & 0.654 & 0.093 \\ 0.424 & 0.285 & 0.116 & 0.245 & 0.075 & 0.093 & 0.071 & 0.079 & 0.246 \end{bmatrix}$$

Step **2**: Compute Covariance (original) Matrix

$$Z = D^T * D = \begin{bmatrix} 0.266061 & 0.439137 & 0.315225 \\ 0.439137 & 1.529584 & 0.562063 \\ 0.315225 & 0.562063 & 0.420554 \end{bmatrix}$$

The **total variance** about the original matrix is = **2.216199** (diagonal summation)

Step **3**: Find eigenvectors using iterative procedure:

$Z \bullet C = E \bullet C$ Use unit vector first which can be derived from the following equation:

$$1 = \mathrm{sqrt}(x^2 + x^2 + x^2) \rightarrow x = \mathrm{sqrt}(1/3) = 0.5773503$$

$$\begin{bmatrix} 0.266061 & 0.439137 & 0.315225 \\ 0.439137 & 1.529584 & 0.562063 \\ 0.315225 & 0.562063 & 0.420554 \end{bmatrix} \bullet \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix} = E \bullet C = \begin{bmatrix} 0.58914 \\ 1.46119 \\ 0.74931 \end{bmatrix}$$

Step **4**: Normalize EC to unit vector and corresponding eigenvalues by dividing each element by the square root of the summation, i.e. sqrt(c1^2+c2^2+c3^2)

$$E1 \bullet C1 = \begin{bmatrix} 0.58914 \\ 1.46119 \\ 0.74931 \end{bmatrix} = 1.744565334 \begin{bmatrix} 0.3377 \\ 0.8375 \\ 0.4295 \end{bmatrix}$$

↑ This is the eigenvector

Step **5**: Iterate unit eigenvector as in previous step

$$\begin{bmatrix} 0.266061 & 0.439137 & 0.315225 \\ 0.439137 & 1.529584 & 0.562063 \\ 0.315225 & 0.562063 & 0.420554 \end{bmatrix} \bullet \begin{bmatrix} 0.3377 \\ 0.8375 \\ 0.4295 \end{bmatrix} = E \bullet C = \begin{bmatrix} 0.59304 \\ 1.67080 \\ 0.75784 \end{bmatrix}$$

Step **6**: Normalize the E•C to unit vector and corresponding eigenvalues:

$$E1 \bullet C1 = \begin{bmatrix} 0.59304 \\ 1.67080 \\ 0.75784 \end{bmatrix} = 1.92810479 \begin{bmatrix} 0.3076 \\ 0.8666 \\ 0.3930 \end{bmatrix}$$

↑ This is updated eigenvector

Step **7**: Iterate unit eigenvector as in previous step

$$\begin{bmatrix} 0.266061 & 0.439137 & 0.315225 \\ 0.439137 & 1.529584 & 0.562063 \\ 0.315225 & 0.562063 & 0.420554 \end{bmatrix} \bullet \begin{bmatrix} 0.3076 \\ 0.8666 \\ 0.3930 \end{bmatrix} = E \bullet C = \begin{bmatrix} 0.58627 \\ 1.68145 \\ 0.74931 \end{bmatrix}$$

Step **8**: Normalize the E•C to unit vector and corresponding eigenvalues:

$$E1 \bullet C1 = \begin{bmatrix} 0.58627 \\ 1.68145 \\ 0.74931 \end{bmatrix} = 1.93195274 \begin{bmatrix} 0.3035 \\ 0.8703 \\ 0.3878 \end{bmatrix}$$

↑ This is updated eigenvector

Step **9**: Solution converges to E1 = 1.93202697 after 8 iterations with first eigenvector:

$$C1 = \begin{bmatrix} 0.30280205 \\ 0.87093411 \\ 0.38702027 \end{bmatrix}$$

Step **10**: The first eigenvalue (E1) accounts for

1.93202697 / **2.216199 = 87.178%**

Step **11**: To calculate second eigenvector and eigenvalue you must first remove the variance associated with the first component from the covariance matrix and obtain a residual variance matrix, RS1:

$$RS1 = Z - E1 \bullet C1 \bullet C1^T$$

$$E1 \bullet C1 \bullet C1^T = 1.93202697 \begin{bmatrix} 0.30280 \\ 0.87093 \\ 0.38702 \end{bmatrix} [0.30280 \quad 0.87093 \quad 0.38702]$$

$$E1 \bullet C1 \bullet C1^T = \begin{bmatrix} 0.1771 & 0.5095 & 0.2264 \\ 0.5095 & 1.4655 & 0.6512 \\ 0.2264 & 0.6512 & 0.2894 \end{bmatrix}$$

This is the variance associated with PC1 subtract from Z

$$RS1 = \begin{bmatrix} 0.266061 & 0.439137 & 0.315225 \\ 0.439137 & 1.529584 & 0.562063 \\ 0.315225 & 0.562063 & 0.420554 \end{bmatrix} - \begin{bmatrix} 0.1771 & 0.5095 & 0.2264 \\ 0.5095 & 1.4655 & 0.6512 \\ 0.2264 & 0.6512 & 0.2894 \end{bmatrix}$$

$$RS1 = \begin{bmatrix} 0.0889 & -0.0704 & 0.0888 \\ -0.0704 & 0.0641 & -0.0892 \\ 0.0888 & -0.0892 & 0.1312 \end{bmatrix}$$

This is left over variance

Step **12**: Now go back to Step **3** and use the RS1 in place of Z:

$$RS1 \bullet C2 = E2 \bullet C2$$

After 8 iterations, the second eigenvector and eigenvalues are

E2 = 0.265419078 and **C2** = $\begin{bmatrix} 0.5398 \\ -0.4914 \\ 0.6835 \end{bmatrix}$

E2 accounts for 0.265419078 /**2.216199 = 11.976%**

E1 + E2 accounts for 99.1538 % of total variance

Step **13**: To calculate third eigenvector and eigenvalue you must remove the variance associated with the first two components, RS2

$$RS2 = Z - E1 \bullet C1 \bullet C1^T - E2 \bullet C2 \bullet C2^T$$

Step **14**: After 8 iterations:

$$\mathbf{E3} = 0.01875295 \quad \text{and} \quad \mathbf{C3} = \begin{bmatrix} 0.7854 \\ 0.0019 \\ -0.6189 \end{bmatrix}$$

E3 accounts for 0.01875295/**2.216199 = 0.846% of total variance**

E1 + E2 +E3 accounts for 100% of total variance

Step **15**: Combine C1, C2, and C3 we arrive at C matrix

Eigenvalues	**Eigenvector**			**Account for**	**Accumulative**
	λ_1	λ_2	λ_3		
1.932027	0.3028	0.8709	0.3870	87.178%	87.178%
0.265419	0.5398	-0.4914	0.6835	11.976%	99.154%
0.018753	0.7854	0.0020	-0.6189	0.8462%	100.00%

Why Use PCT?

- *Decorrelates the spectral data*
- *Multispectral bands are often highly-correlated because of:*
  - *Material spectral correlation*
  - *Topography*
  - *Sensor band overlap*

Why Not Use PCT?

- *Data--dependent*
  - *W coefficients change from scene--to-scene*
  - *Makes consistent interpretation of PC images difficult*
- *Spectral details, particularly in small areas, may be lost if higher-order PCs are ignored*
- *Computationally expensive for large images or for many spectral bands*

Tasseled Cap Component

- *Linear spectral transform like the PCT*

$$TC = W_{TC} \bullet DN$$

- *In this case, the W matrix is fixed for a given sensor*

Tasseled Cap Component

- *Table 5-2 Tasseled-cap components for MSS and TM*

sensor	name	w_{TC}			
	MSS band	1	2	3	4
L-1 MSS	soil brightness	+0.433	+0.632	+0.586	+0.264
	greenness	–0.290	–0.562	+0.600	+0.491
	yellow stuff	–0.829	+0.522	–0.039	+0.194
	non-such	+0.223	+0.120	–0.543	+0.810
L-2 MSS	soil brightness	+0.332	+0.603	+0.676	+0.263
	greenness	+0.283	–0.660	+0.577	+0.388
	yellow stuff	+0.900	+0.428	+0.0759	–0.041
	non-such	+0.016	+0.428	–0.452	+0.882

TM band	1	2	3	4	5	7

Tasseled Cap Component

- ***Table 5-2 Tasseled-cap components for MSS and TM***

sensor	name	w_{TC}					
L-4 TM	soil brightness	+0.3037	+0.2793	+0.4743	+0.5585	+0.5082	+0.1863
	greenness	–0.2848	–0.2435	–0.5436	+0.7243	+0.0840	–0.1800
	wetness	+0.1509	+0.1973	+0.3279	+0.3406	–0.7112	–0.4572
	haze	–0.8242	+0.0849	+0.4392	–0.0580	+0.2012	–0.2768
	TC5	–0.3280	+0.0549	+0.1075	+0.1855	–0.4357	+0.8085
	TC6	+0.1084	–0.9022	+0.4120	+0.0573	–0.0251	+0.0238
L-5 TM	soil brightness	+0.2909	+0.2493	+0.4806	+0.5568	+0.4438	+0.1706
	greenness	–0.2728	–0.2174	–0.5508	+0.7221	+0.0733	–0.1648
	wetness	+0.1446	+0.1761	+0.3322	+0.3396	–0.6210	–0.4186
	haze	+0.8461	+0.0731	+0.4640	–0.0032	–0.0492	+0.0119
	TC5	+0.0549	–0.0232	+0.0339	–0.1937	+0.4162	–0.7823
	TC6	+0.1186	–0.8069	+0.4094	+0.0571	–0.0228	+0.0220
	soil brightness	additive terms:	+10.3695				
	greenness		–0.7310				
	wetness		–3.3828				
	haze		+0.7879				
	TC5		–2.4750				
	TC6		–0.0336				

Tasseled Cap Component

- **Landsat 7** Huang et al., 2002

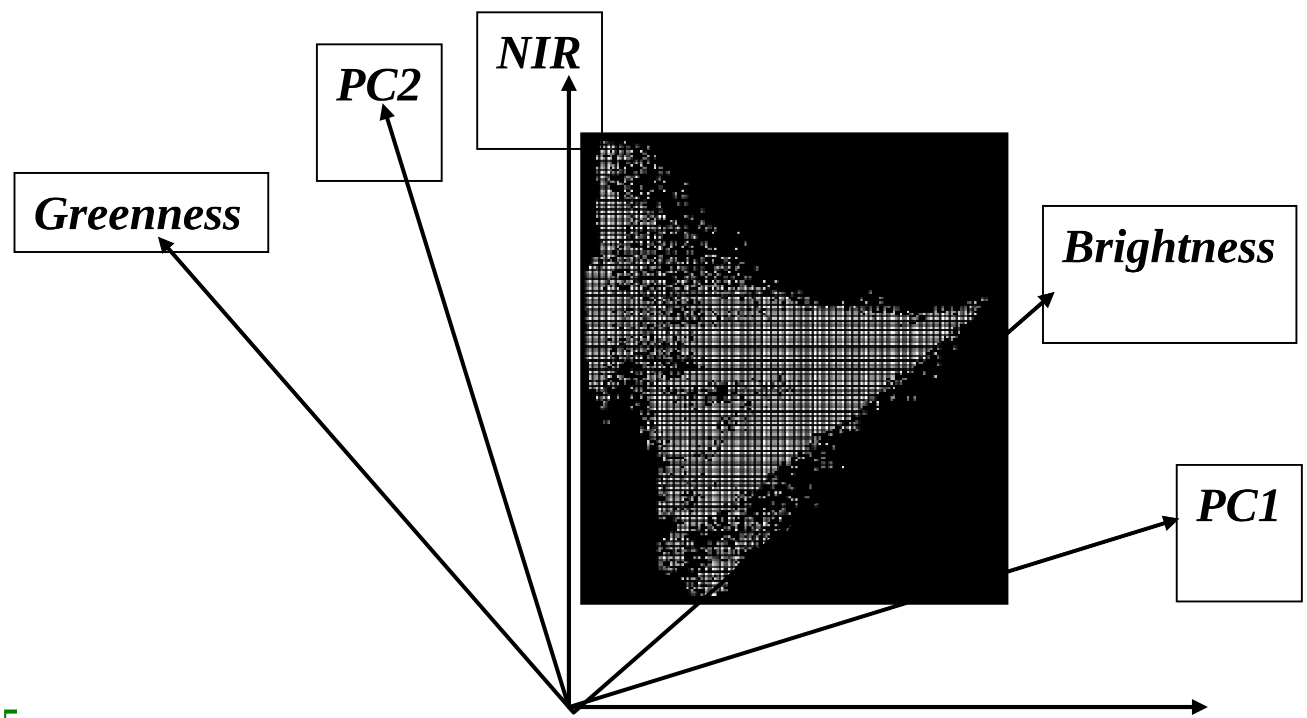
0.3561	0.3972	0.3904	0.6966	0.2286	0	0.1596
-0.3344	-0.3544	-0.4556	0.6966	-0.0242	0	-0.2630
0.2626	0.2141	0.0926	0.0656	-0.7629	0	-0.5388
0.0805	-0.0498	0.1950	-0.1327	0.5752	0	-0.7775
-0.7252	-0.0202	0.6683	0.0631	-0.1494	0	-0.0274
0.4000	-0.8172	0.3832	0.0602	-0.1095	0	0.0985

Why Use the TCT

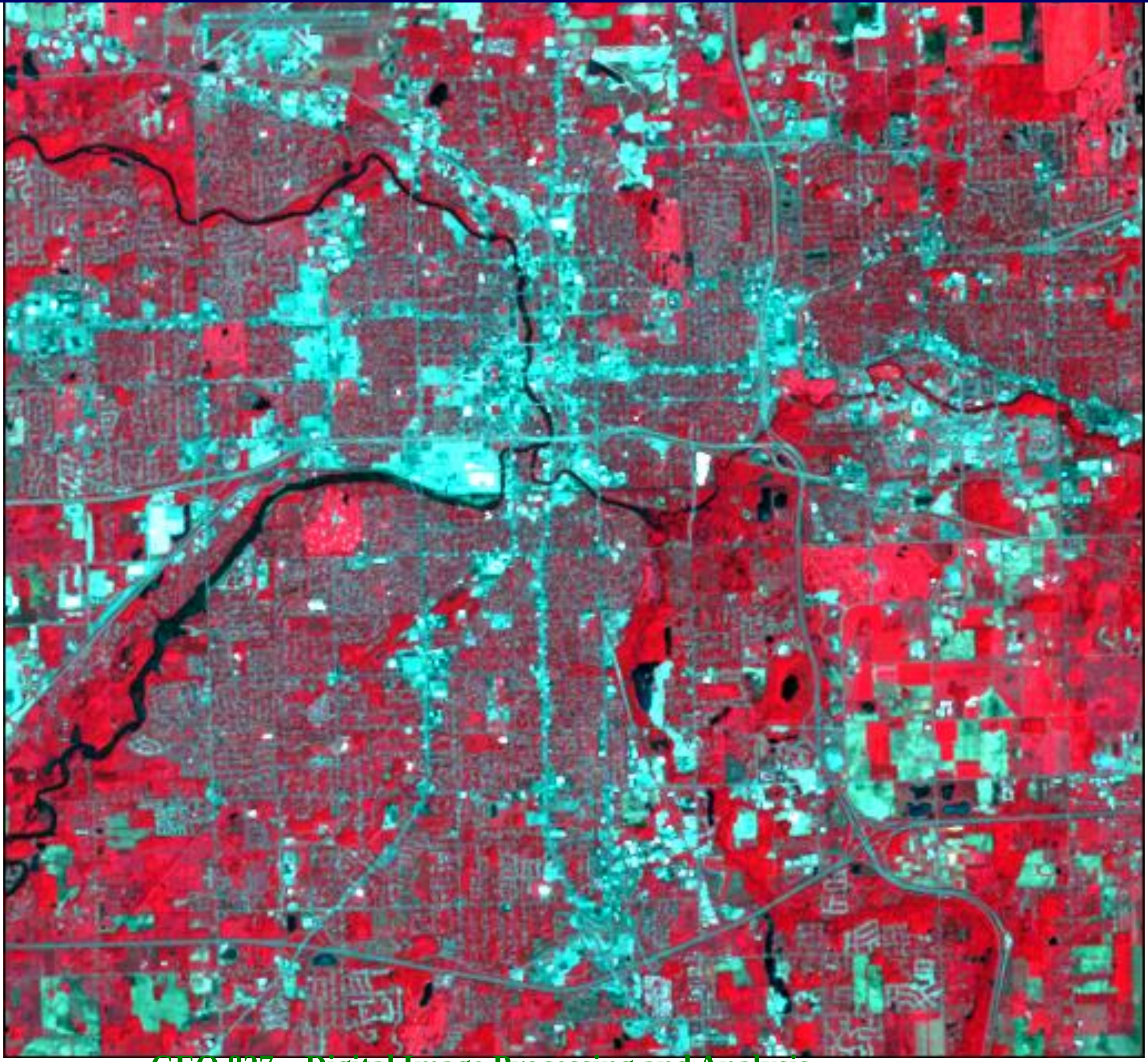
- *It is a fixed reference,related to geophysical properties of the scene*
  - *First component is "soil brightness "*
  - *Second component is "greenness "*
  - *Third component is "yellowness" or "haze" or "wetness"*
  - *Forth component is "non-such"*
- *It was also referred as "n-space" index*

Why Not Use the TCT

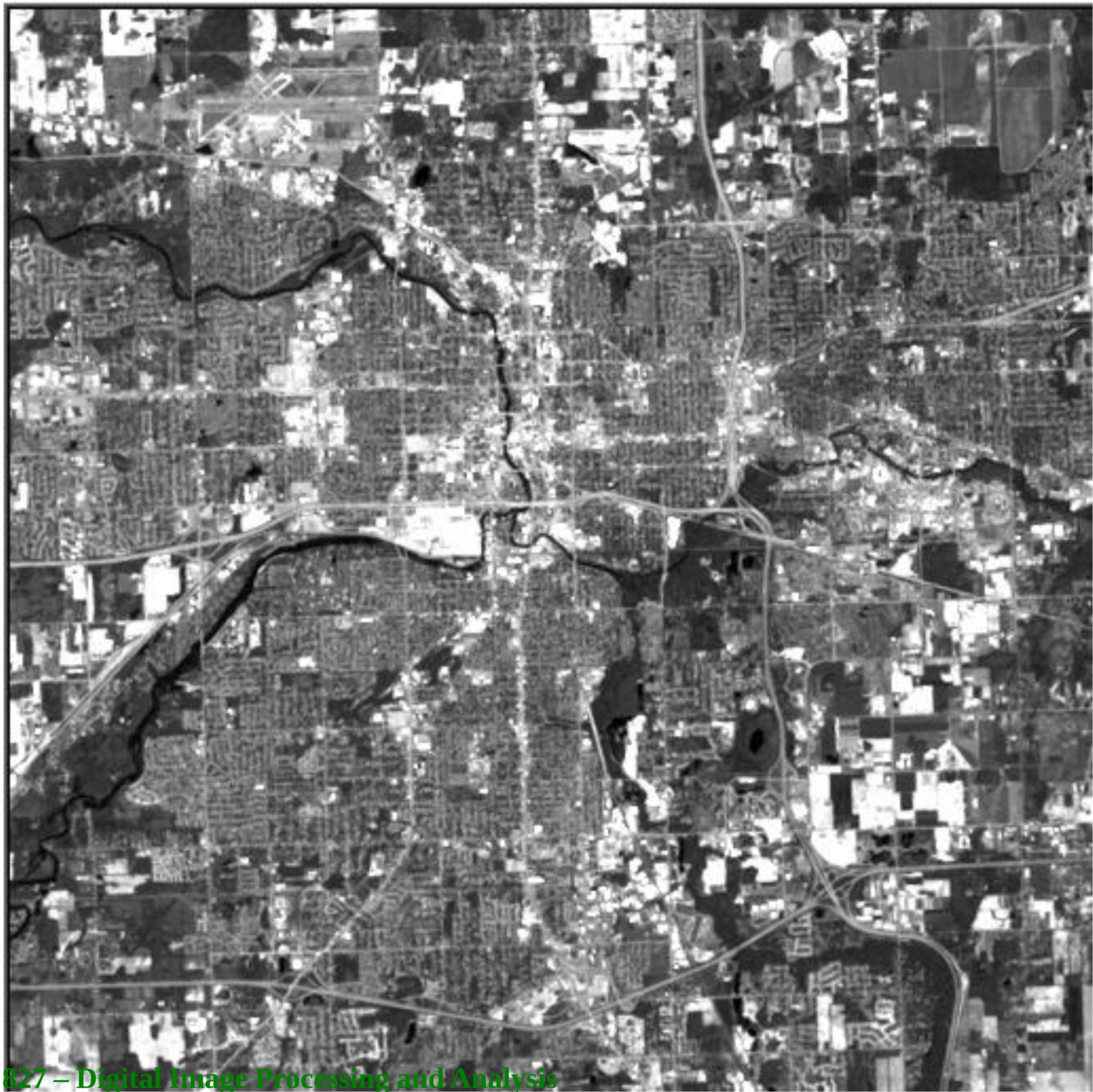
- *Nonoptimal compression of data*
- *Requires multitemporal data for each sensor to derive W_{TC}*



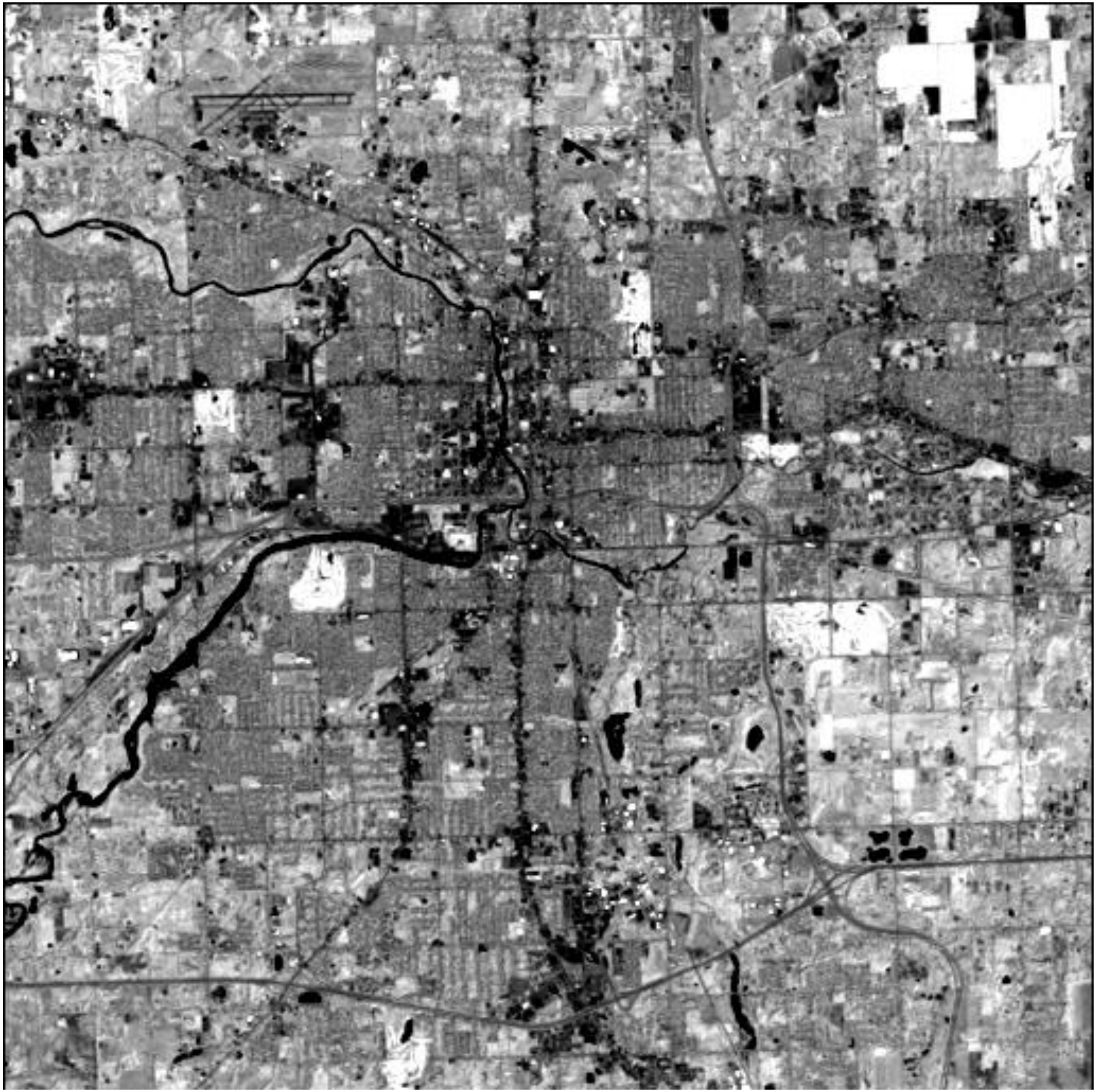
Landsat 7 ETM image over Lansing



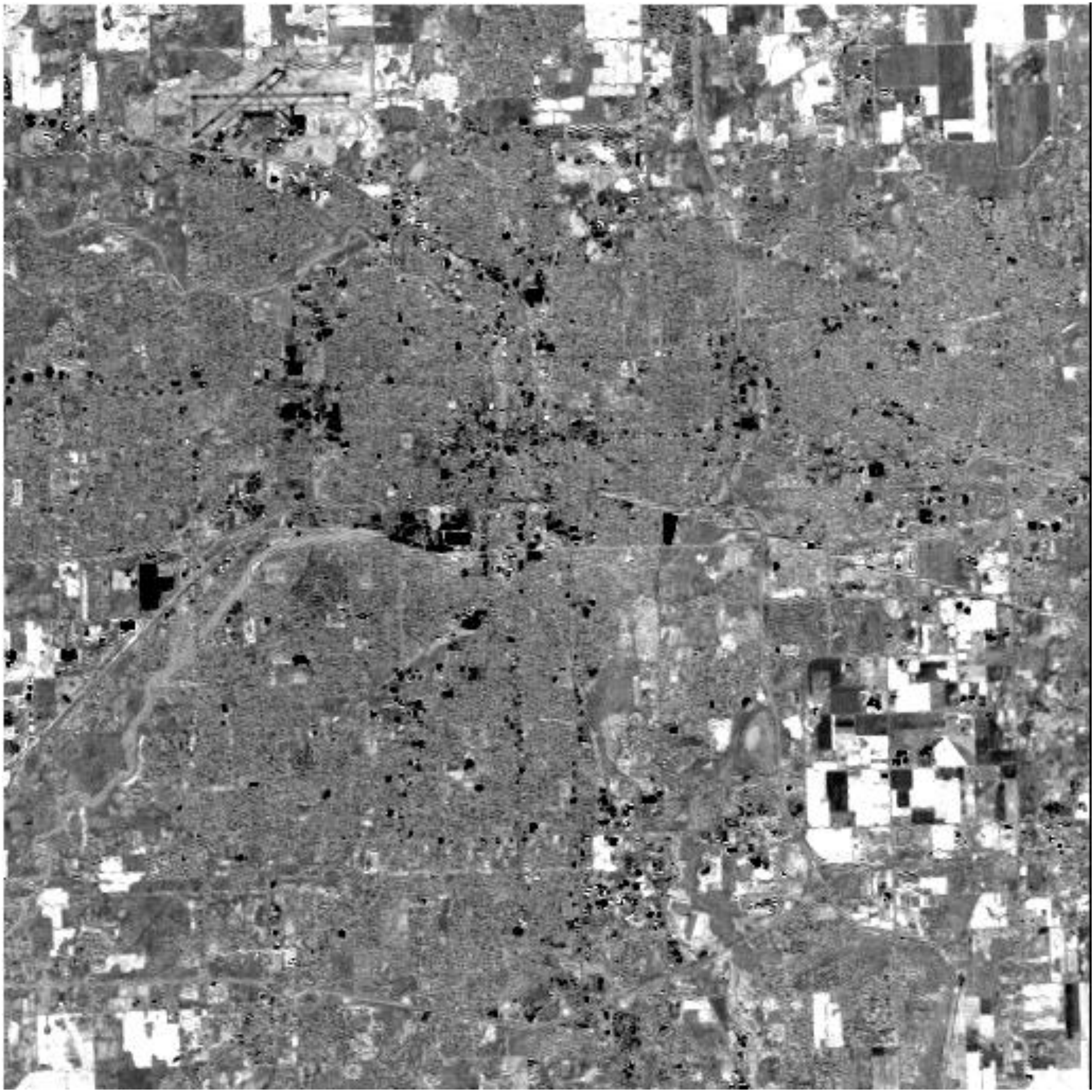
First PC1



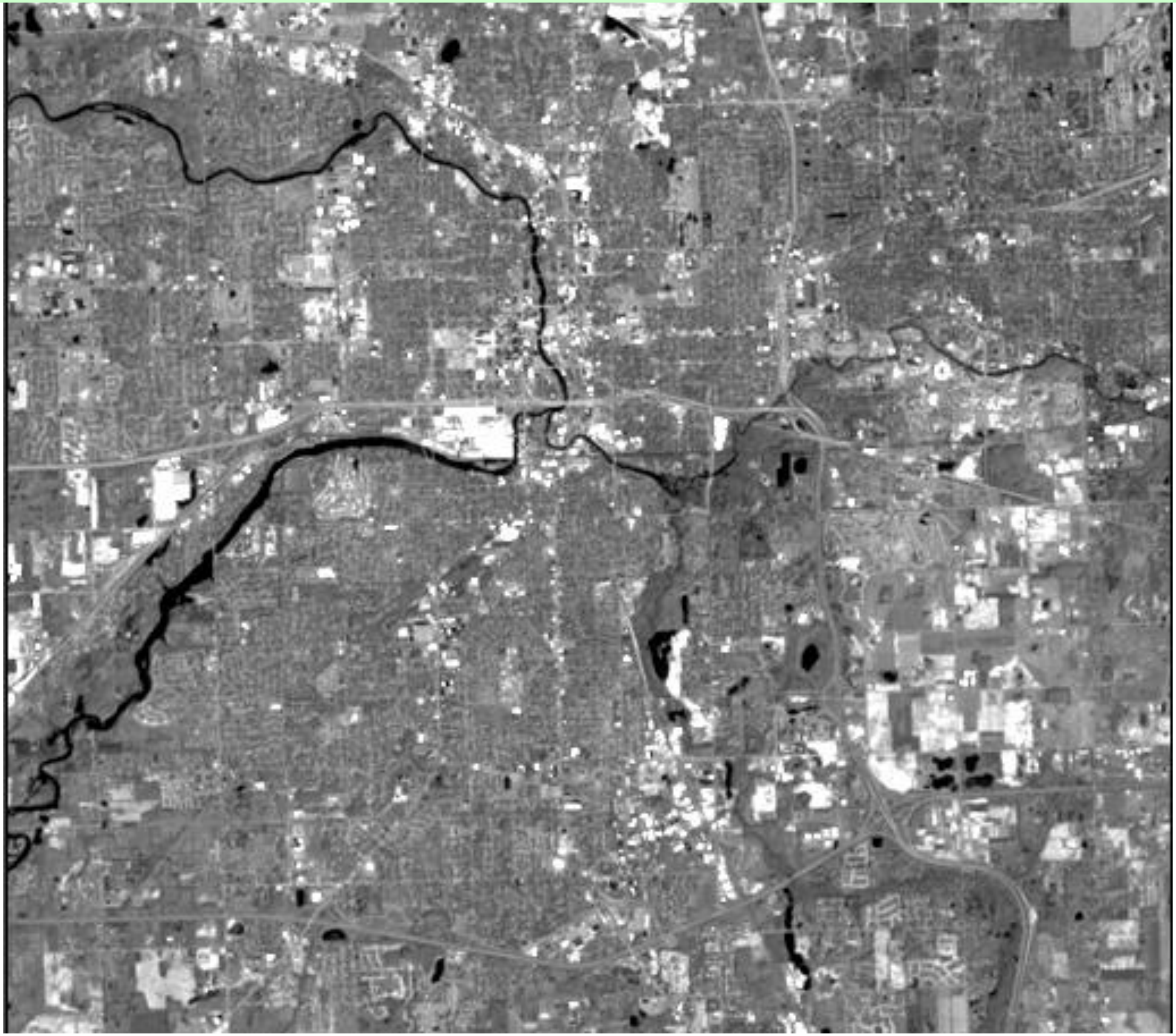
2nd PC2



3rd PC3



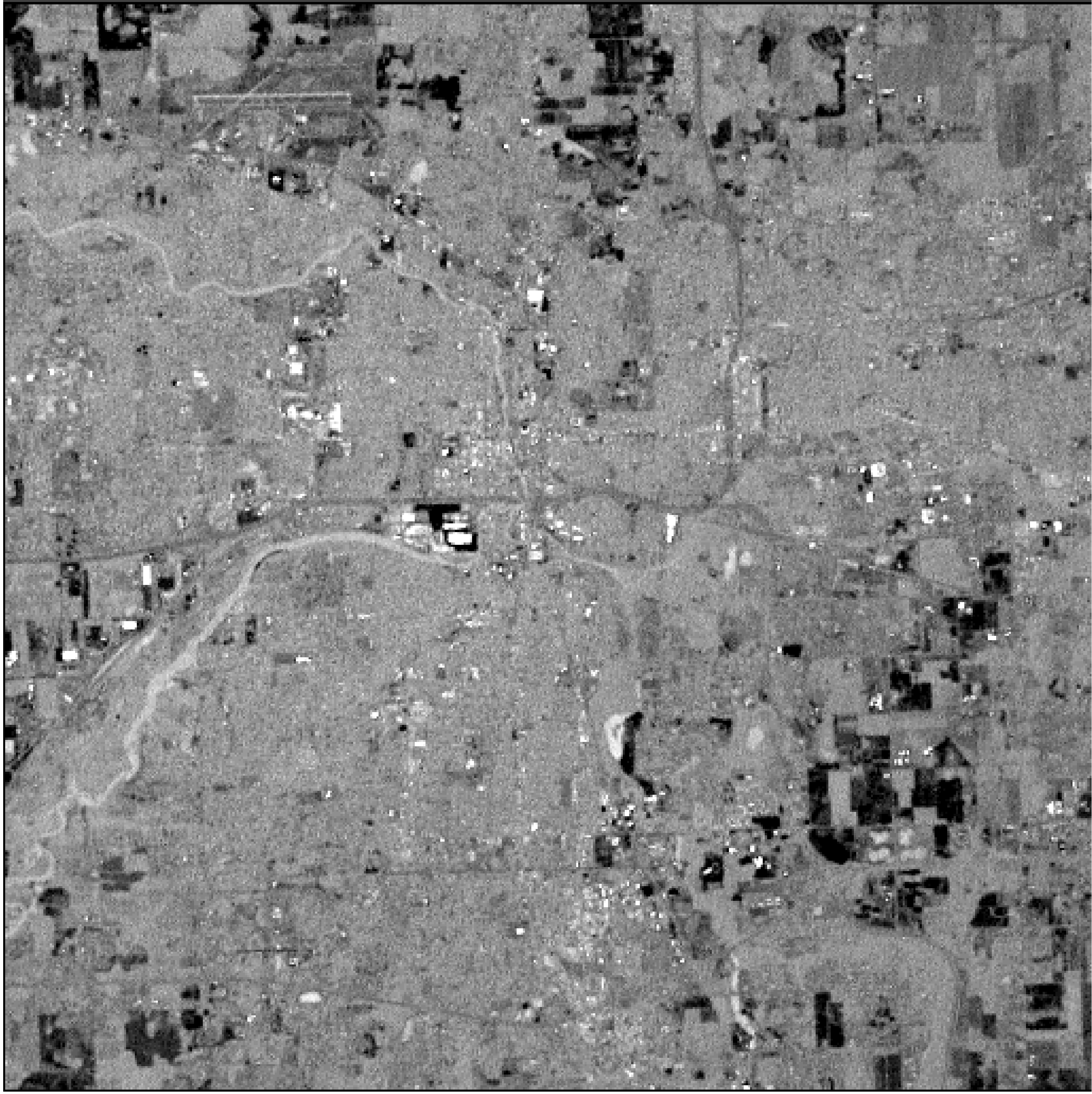
First TC1 – Soil Brightness



Second TC2 - Greenness

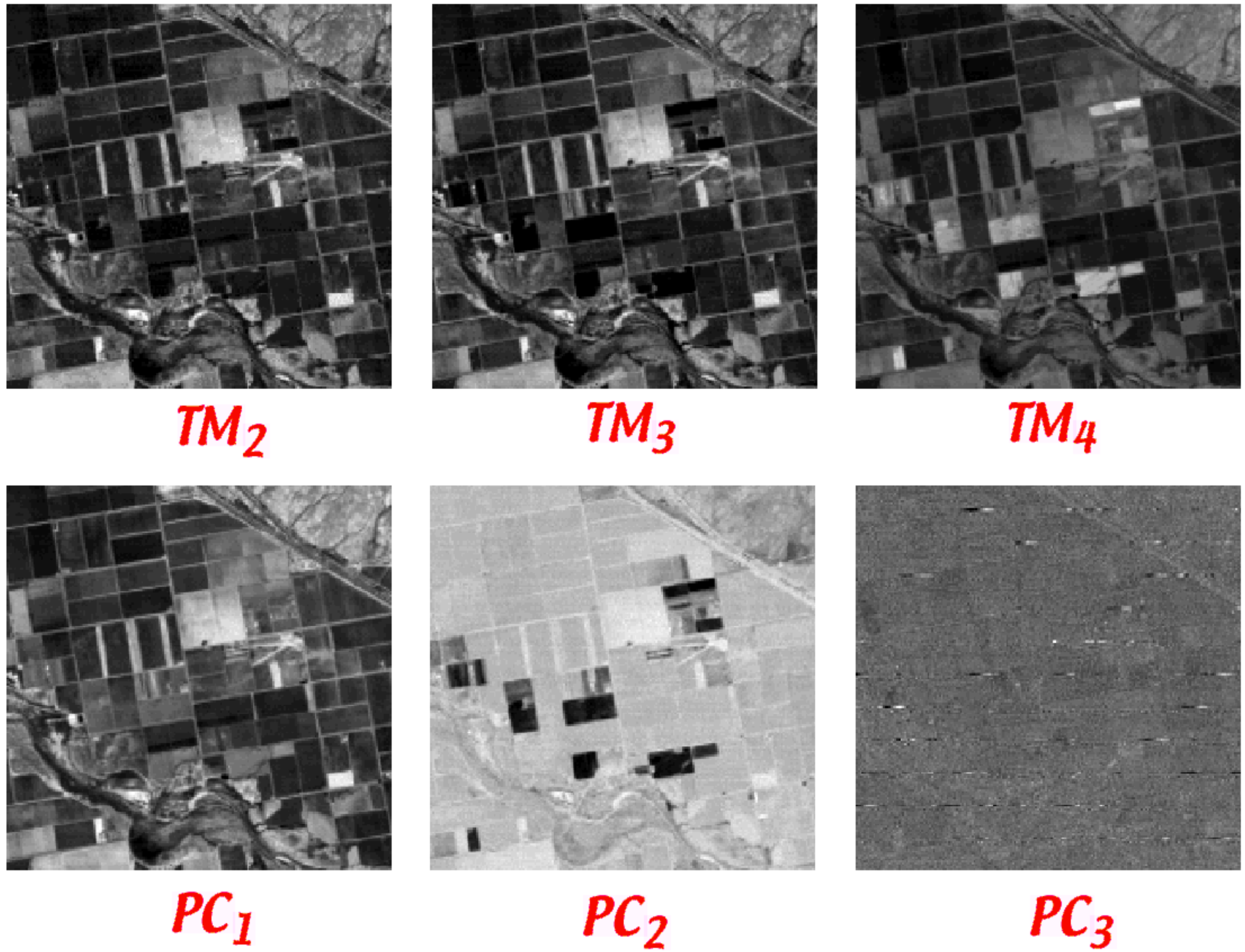
Third TC3 – Yellowness

Fourth TC4 – Non-such



Noise Detection with PCA

- Noise detection by spectral correlation
- Spectrally-uncorrelated noise is isolated by PCT



From Schowengerdt, p298

Other Transforms

- SPC – Standard Principal Component
  - Based on correlation, rather than covariance, matrix
- MNF – Maximum Noise Fraction
  - Also known as the Noise-Adjusted Principal Components. It was the modification of the PCT and meant to improve the isolation of image noise that may occur in one or only a few spectral bands.